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A mathematical analysis of Monson's spherical theory and its clinical implications

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ABSTRACT

Statement of Problem: The need to subject Monson's spherical theory to further mathematical investigation is imperative in view of its clinical importance in dental occlusion.

Purpose: The main goal of this study was to test the following hypotheses:

1. Monson's pyramid is the 3D unifying geometric figure for Bonwill's, Spee's, Monson's, and Hall's theories of occlusion.

2. Monson's sphere is made up of four regular tetrahedrons (Monson's pyramids).

3. The radius of Monson's sphere is greater than the radius of circumsphere of Monson's pyramid.

Materials and Methods: Bonwill's triangle was used as the basis of geometrical model for constructing other 3D objects in this study; and it was assumed that the length of each side of Bonwill's triangle was 10cm. A regular tetrahedron was constructed from Bonwill's triangle. Then, linear and angular parameters were calculated for the constructed tetrahedron and its associated geometric figures. The calculated values were then subjected to statistical analysis using SPSS version 20; and comparisons of parameters were made using student's t-test.

Results: It was found that the theoretical geometrical figures that were proposed and demonstrated by Bon will, Spee, Monson and Hall were interconnected geometrically by means of a 3D geometric figure known as tetrahedron.

Conclusion: Monson's pyramid was established as the unifying 3D geometric figure for the analyzed geometric models of occlusion. Monson's sphere is made up of four Monson's pyramids while the radius of Monson's sphere is also found to be greater than the radius of circumsphere of Monson's pyramid. The clinical significance of this study is that some important linear and angular parameters, that are required in the fabrication of dentures, can be calculated from a regular tetrahedron and its associated geometric figures based on individual patient's bicondylar distance.

Keywords: Connectivity, Mathematics, Occlusion, Theory.

Introduction

Many mathematical models have been developed to describe dental occlusion as well as to explain different occlusal concepts; and the importance of these mathematical models cannot be overlooked because of their contributions in ensuring precision in the processes and principles of bioengineering design and construction of dental appliances, instruments and equipment [1-4]. Mathematical models are aimed at structural stability, integrity, functional effectiveness, efficiency and reliability. Structural designs are based on three fundamental geometric figures to wit: circle, triangle and square; and all other geometric figures are derivatives of these primary geometric figures.

The initial work on mathematical model for occlusal concepts was pioneered by Bonwill [1]. In 1846, Bonwill [1] proposed that mandibular movements were guided by the condylar and incisal guidance; and the teeth moved in relation to each other as guided by this guidance. He advanced that the joining of the two condylar processes of the mandible and the midpoint of the medial contact of lower central incisors with lines, an equilateral triangle was formed; and each side of the triangle was approximately 10cm. This is known as Bonwill's equilateral triangle and it forms the basis of construction of some average articulators.

In 1890, Von Spee [2,3] proposed that during protrusive movement of the mandible, the condyles and the teeth describe a common curve which was a part of a circle with the centre at the glabella; and the radius of the circle was about 10cm. This curve is referred to as sagittal occlusal curve and lies at the lower segment of the circle.

In 1918, Monson [4-6] linked up Spee's curves with perpendicular bisectors of their whole chords and demonstrated that the bisecting lines intersected at the glabella. He also demonstrated that the resultant force from the lower teeth also intersected at the glabella. This point of intersection is the centre of a sphere having a radius of about 10cm; and its lower segment represents the occlusal surface. He also demonstrated the presence of lateral occlusal curves owing to the different levels of the lingual and buccal cusps of posterior teeth. Hall [7,8] proposed that the lower teeth move over the surfaces of the upper teeth as over the surface of a cone, generating an angle of 45-degrees with the central axis of the cone tipped at 45 degrees to the occlusal plane. The base of the cone lies posterior while its apex is the midpoint of the lower central incisors. The studies of Bonwill [1], Spee [2,3], Monson [4,5] and Hall [7,8] were based on the premise that the left and right sides of the mandible were symmetrical. However, studies by Bosse [9], Choquet [10,11], Welcher [12], Frahm [13] Amoedo [14] and Wilson [15] have shown high degree of asymmetry of the left and right sides of the mandible. Variations in the bicondylar distances among people within a racial grouping as well as between racial groupings of different geographical locations have also been observed and reported [9-15].

The concepts of occlusion and articulation have been subjected to a long-standing debate and exposition by dentists; and there are still few grey areas on which no common ground has been reached. This topic will continue to attract new studies because it is central to diagnosis, treatment planning and treatment evaluation in the different fields of dentistry [16-18].

The following hypotheses were tested in this study:

1. Monson's pyramid, a regular tetrahedron derived from Bonwill's triangle, is the 3D unifying geometric figure for Bonwill's, Monson's, Spee's and Hall's theories of occlusion.
2. Monson's sphere is made up of four Monson's pyramid, a regular tetrahedron derived from Bonwill's triangle.
3. The radius of Monson's sphere is greater than the radius of circumsphere of Monson's pyramid.

Materials and Methods

This study was a mathematic-geometric analysis of the relationships among Bonwill's triangle, Spee's circle/curves, Monson's sphere and Hall's cone. Bonwill's triangle was used as the basis of geometric construction of the other geometric models as well as generation of linear and angular numerical data for this study. It was assumed that the length of each side of Bonwill triangle was 10cm. The following parameters to wit: radius, area, volume, height, length of side, length of edge and angles were determined, as the case might be, in respect of Bonwill's triangle, Monson's pyramid, Spee's circle, Monson's sphere and Hall's cone using various mathematical formulae as shown in tables 1a and 1b.

List of tables and figures

P=3a	*Perimeter*
S=3a÷2	*Semi-perimeter*
A= a2√3÷4	*Area*
H= a√3÷2	*Altitude*
H= a√3÷2	*Median*
H= a√3÷2	*Angle Bisector*
R= a√3÷3	*Circumscribed Circle Radius*
r = a√3÷6	*Inscribed Circle Radius*

Table 1a: Showing equilateral triangle equations [21].

Surface area	*A=* $a^2\sqrt{3}$
Volume	*V= 1/12* $a^3\sqrt{3}$
Height	*h= √6(a/3)*
Angle between an edge and a face	*Arctan (√2)* *(approx. 55°)*
Angle between two faces	*arccoc(1/3)= arctan(2 √2)* *(approx. 71°)*
Radius of circumsphere	*R= √6(a/4)*
Radius of insphere that is tangent to faces	*r= √6(a/12)*

Table 1b: : Showing some formulas for regular tetrahedron [26], For a regular tetrahedron of edge length 'a'.

Mathematic-geometric formation of Bonwill's triangle, Spee's circle, Monson's sphere and Hall's cone

The interconnectedness of Bonwill's triangle, Spee's circle, Monson's sphere and Hall's cone were carried out as follows: Concerning Bonwill's triangle, the right and left sides of the triangle were regarded as the right and left extended chords of Spee's curves. The perpendicular bisectors from all the sides of Bonwill's equilateral triangle also intersected at the point similar to the common centre of rotation of Spee's circle and Monson's sphere respectively. Therefore, perpendicular lines, drawn upward, from the three vertices of the Bonwill's triangle would result in the formation of a regular tetrahedron since all the three sides are of the same length (Figure 1).

The tetrahedron could also be regarded as a cone or a pyramid with a triangular base in line with Hall's cone (Figure 1).

Concerning Spee's curves/circle, the right and left sides of Bonwill's triangle were regarded as the extended right and left chords of Spee's curves. The perpendicular bisectors from these whole chords of Spee's curves on both sides of the mandible intersect at one point. If a whole chord and its perpendicular bisector of one side of Spee's curves occupy a two-dimensional space, then, the perpendicular bisectors of the two whole chords of Spee's curves on both sides of the mandible intersect at one point, and this point displays a three dimensional space location. This is the rotation centre of a sphere. Therefore, a similar geometric construction can be applied to produce a sphere having a common centre of rotation with Monson's sphere, as previously described based on Spee's curves (Figure 1). Linear and angular parameters of Bonwill's triangle, Spee's circle/curves, Monson's sphere,

Hall's cone and the derived regular tetrahedron were calculated using mathematical and geometrical formulae as shown in tables 1a and 1b.

The data obtained were analyzed using SPSS version 20. Comparisons of linear parameters concerning the geometric figures were carried out using student's t-test. Level of significance was set at p≤0.05 and confidence interval was set at 95%.

Results

The regular tetrahedron derived from Bonwill's triangle constitutes the unifying geometric figure for Bonwill's triangle, Spee's curves/circle, Monson's sphere/curves and Hall's cone as shown in figure 1.

The radii of incircle and circumcircle of Bonwill's triangle are 2.88675135cm and 5.7735027cm respectively; and they have a common centre (Table 2). The radii of insphere and circumsphere of Monson's pyramid or tetrahedron are 2.041 cm and 6.123cm respectively; and they share a common centre (Table 2). The radius of the circumcircle of Bonwill's triangle is twice the radius of its incircle (Figure 3). The radius of insphere of a regular tetrahedron is ¼ of the height of the tetrahedron while the radius of the circumsphere of the tetrahedron is thrice the radius of insphere of the tetrahedron (Tables 2 and 3). The volume of Monson's sphere is 4190.47619 cm3 (Table 3) while the volume of the circumsphere of the regular tetrahedron (Monson's pyramid) is 961.0958cm3 (Table 3) with a ratio of 4:1 approximately.

The areas of Bonwill's triangle, Spee's circle, Monson's sphere, the regular tetrahedron (Monson's pyramid) and Hall's cone are 43.3cm2, 173.2cm2, 1257.14286cm2, 314.285714cm2 and 173.2cm2 respectively (Table 3).

The length of each side of Bonwill's triangle and the length of each edge of the regular tetrahedron (Monson's pyramid) and Hall's cone are the same (10cm). The heights of Bonwill's triangle, Monson's pyramid, Monson's sphere, Spee's circle and Hall's cone are 8.66025404cm, 8.16496581cm, 20cm, 20cm and 8.16496581cm respectively (Table 3). Table 4 shows some important values of the regular tetrahedron, Bonwill's triangle, Balkwill's angle and Hall's cone. The length of a side of an equilateral triangle that will perfectly fit into Spee's circle is 17.3205081cm and this is greater than the length of one side of Bonwill's triangle i.e. 17.3205081cm-10.0cm =7.3205081cm (Table 5). The length of an edge of a regular tetrahedron that will perfectly fit into Monson's sphere is 16.3299316cm and this is greater than the length of one side of Bonwill's triangle i.e. 16.3299316cm-10.0cm=6.3299316cm (Table 5). No significant difference between the length of an edge of tetrahedron that will perfectly fit into Monson's sphere and the length of one side of the equilateral triangle that will perfectly fit into Spee's circle provided the length of one side of Bonwill's triangle is 10cm (P=0.154) (Table 5).

No significant difference between the radius of the circumsphere of Monson's pyramid and the radius of circumcircle of Bonwill's triangle provided the length of one side of Bonwill's triangle is 10cm (P=0.432) (Table 5).

Parameters	Incircle and circumcircle of Bonwill's Triangle		Insphere and circumsphere of Monson's Pyramid	
	Incircle	Circumcircle	Insphere	Circumsphere
Radius (cm)	2.886	5.772	2.041	6.123
Area (cm2)	26.176	104.852	52.368	471.317
Volume (cm3)	-	-	35.368	961.958
Height (cm)	5.772	11.544	4.082	12.246

Table 2. Comparison of parameters of incircle and circumcircle of Bonwill's triangle as well as insphere and circumsphere of Monson's pyramid.

Parameters	Bonwill's triangle	Monson's pyramid r = 6.123	Monson Sphere rrr r=10cm	Spee's circle/curve r r=10cm	Hall's cone
Height (cm)	8.66025404	8.16496581	20.0	20.0	8.16496581
Area (cm2)	43.3	173.2	1257.14286	314.285714	173.2
Volume (cm3))	-	13.576667	4190.47619	-	13.5766667
Length of edge/side (cm)	10.0	10.0	-	-	10.0

r = radius.

Table 3. Comparison of parameters of Bonwill's triangle, Monson's pyramid, Monson's sphere, Spee's circle and Hall's cone.

Parameters	Monson's Pyramid, Bonwill's Triangle, Balkwill's Triangle & Hall's Cone
Angle between Bonwill's triangle and the occlusal plane	Balkwill angle =26o(mean)
The three internal angles formed at the vertices of Bonwill's triangle	facial triangle =60o
Angle between a surface of the tetrahedron and one of its edges	inclination angle =70.5o
Angle between the central axis of the cone and a face	19.47o
Halls inclination angle i.e. the tilt of the cone in relation to occlusal plane =45o	26o +19.47o = 45.47o Balkwill's angle (26o)+ face-axis angle (19.47o) = 45.47o

Table 4. Some important angular values of Monson's pyramid (Tetrahedron derived from Bonwill's Triangle), Bonwill's triangle, Balkwill's angle and Hall's cone.

Parameters		Difference	P-value
The difference between length of a side of equilateral triangle that will perfectly fit into .Spee's circle and length of one side of Bonwill's equilateral triangle	7.320508	0.990	0.154
The difference between length of an edge of regular tetrahedron that will perfectly fit into Monson's sphere and length of one side of Bonwill's triangle	6.3299316		
The difference between radius of circumcircle of Bonwill's triangle and radius of Spee's circle	4.22643	0.339	0.432
The difference between radius of circumsphere of Monson's tetrahedron and radius of Monson's sphere	3.887		

Table 5. Comparison of radius and length parameters of Bonwill's triangle, Spee's circle and Monson's sphere/tetrahedron.

Discussion

The discussion of this study would be based on the following five major sections:

i. Interconnectedness of Bonwill's triangle, Spee's circle/curve, Monson's sphere/curves, and Hall's cone.

ii. The relationship between the radius of a circle and the length of its inscribed equilateral tringle.

iii. The relationship between the radius of a sphere and the length of the edge of its inscribed regular tetrahedron (triangular pyramid).

iv. Circumscribed regular tetrahedron (derived from Bonwill's triangle) and its roles in occlusal concept.

v. Determination of Hall's cone inclination.

Interconnectedness of Bonwill's triangle, Monson's sphere/curves, Spee's circle/curves and Hall's cone

This study shows a strong connection among the analyzed theoretical geometric models of occlusion. The strong inter-connectedness is rooted in the trajectory of the mandible, as guided by the condylar processes and the incisal guidance, during mandibular excursions [1-4]. Bonwill's equilateral triangle laid the foundation upon which Spee's circle/curves, Monson's sphere/curves/tetrahedron and Hall's cone were developed as shown in figure 1. The interconnectedness of the aforementioned geometrical models of occlusion is further demonstrated by the common centre of rotation of Spee's circle/curves, Monson's sphere/curves/pyramid and Hall's cone which were derived from Bonwill's equilateral triangle (Figure 1)) [19-23].

Hall's cone does share the same center of rotation with Spee's circle and Monson's sphere but it is different in terms of the orientation of its long axis to the occlusal plane. Hall's cone is the same regular tetrahedron derived from Bonwill's triangle but its apex lies at the medial contact of the lower central incisors while its base lies posteriorly (Figure 1). The regular tetrahedron (triangular pyramid), formed from Bonwill triangle, through the work of Monson, could be regarded as the geometric link that could adequately unify and describe these geometrical models. It should be noted that a regular tetrahedron is formed from four equilateral triangles (Figure 1); and every equilateral triangle has an incircle, a circumcircle and an excircle (figure 2) [19-23]. In a similar state, every tetrahedron is associated with an insphere, a circumsphere, a midsphere and an exosphere (Figure 3) [19-23]. Thus, an equilateral triangle and a tetrahedron share similar geometric characteristics owing to the fact that the regular tetrahedron is derived from four equilateral triangles. Consequently, theories and concepts of occlusion, as propounded by Bonwill, Spee, Monson and Hall, are closely interrelated based on their genesis. In this context, it could be argued that all theories and concepts of occlusion are partly rooted in past research findings by notable authors in the field of human occlusion [1-8].

The relationship between the radius of a circle and the length of its inscribed equilateral tringle.

The length of the equilateral triangle that would perfectly fit into Spee's circle with a radius of 10cm was 17.3205081cm. This length was significantly longer than the length of a side of Bonwill's triangle which is 10cm. In this study, it was also found that the radius of the circumcircle of Bonwill triangle was 5.77357cm; and this circumradius was significantly shorter than the radius of Spee's circle which is 10.0cm.

This suggests that the length of one side of a circumscribed equilateral triangle is directly proportional to the radius of its circumcircle [19]; then, the questions arise: under what situation will Spee's circle, with a radius of 10cm, accommodate Bonwill triangle; and how many such triangles can be accommodated within the circle? The answers to these questions lie in the fundamental facts of geometric theorems that if a chord of 10cm subtends an angle of 60^{o} at the centre of a circle with a radius of 10cm, an equilateral triangle which is equivalent to Bonwill's triangle is formed. The total angle at the centre of a circle (angle at a point) is 360 degrees. Therefore, six equilateral triangles will subtend a total of 360 degrees at the centre of the circle i.e.$360^{o} \div 60^{o} = 6$. Consequently, Spee's circle with a radius of 10cm will be made up of six equilateral triangles; and each triangle will have one of its sides as a chord on a portion of the circumference of the circle and the other two sides as the radii that subtend an angle of 60 degrees at the centre of the circle. Consequently, the Spee's circle thus gives an appearance of a wheel having six sectors; and the arc that is formed by one of the sides of the lowest equilateral triangle represents the sagittal occlusal curve of Spee (Figure 4).

The Spee's circle can also be likened to a circumscribed six-sided regular polygon (Figure 4) in which each side subtends an angle of 60 degrees at the centre of the circle; and the radius of the circumcircle and the length of each side of the polygon are the same (10cm). The circumcircle of the individual equilateral triangle within the Spee's circle has a radius of 5.773570cm. Therefore, the length of side of the equilateral triangle that is circumscribed by a circle with a radius of 10cm is significantly longer than the length of a side of Bonwill's triangle.

The relationship between the radius of a sphere and its inscribed tetrahedron

Another important finding of this study was that the length of edge of the tetrahedron that would perfectly fit into Monson's sphere with a radius of 10cm was 16.3299316cm. This length is significantly longer than the length of a side of Bonwill's triangle which is 10cm. It was also found that the radius of the circumsphere of Monson's pyramid was 6.123cm; and this circumradius was significantly shorter than the radius of Monson's sphere which is 10cm.

Therefore, this suggests that the length of edge of any circumscribed regular tetrahedron is determined by the radius of its circumsphere. Then, one may ask, under what situation can Monson's sphere accommodate a tetrahedron whose length of edge will be equivalent to the length of a side of Bonwill's triangle; and how many such tetrahedron can be accommodated within the Monson's sphere? Geometrically, the tetrahedron whose length of edge is equivalent to Bonwill triangle is considered as a sector of Monson's sphere and has its base located at the surface of the sphere while its apex is located at the centre of the sphere (Figure 1).

If the volume of the circumsphere of Monson's tetrahedron is 961.9581cm^3 and the volume of Monson's sphere is 4190.47619cm (Table 3). Then, the number of circumscribed Monson's pyramids that will fill Monson's sphere is equal to the volume of Monson's sphere divided by the volume of the circumscribed Monson's pyramid (4190.47619cm^3 ÷ 961.9581cm$^{3)}$ = 4.0 approximately. Therefore, the length of edge of the tetrahedron that is circumscribed by Monson sphere with a radius of 10cm is significantly longer than the length of a side of Bonwill's triangle.

The circumscribed Monson's pyramid (a regular tetrahedron) and its roles in occlusion concepts

The roles of circumscribed Monson's pyramid in occlusal concepts and occlusion are clearly shown in this study. The circumscribed Bonwill's triangle portion of the Monson's pyramid represents a cross section of Monson's sphere and its radius is 5.773570cm which is significantly smaller than the radius of Monson's sphere (Figure 1 & Table 3). Therefore, it does suggest that the plane formed by Bonwill's triangle does not lie at the centre of the sphere (Figure 1). However, the areas of the segments formed by left and right sides of Bonwill's triangle with its circumcircle represent the surface areas occupied by the combined medio-lateral surfaces of the upper and lower teeth (Figure 2). These areas constitute the boundaries of the medio-lateral movements of the mandible. The distance between the right or left side of Bonwill's triangle and the arc of the segment formed by either the right or left side decreased posteriorly and anteriorly from the mid-point of the curvature (arc) owing to the decrease in the medio-lateral width of the teeth either posteriorly or anteriorly.

The lateral and sagittal curves intersect at the surface of the sphere to form Monson's curves (Figure 1). Therefore, it could be appropriately suggested that lateral and sagittal curves are parts of Monson's sphere but they exhibit different orientations with respect to the three major planes of occlusion and articulation (Figure 1). Spee's curves [2,3] run in anterior-posterior plane (sagittal plane), therefore they are called sagittal occlusal curves while Wilson's curves [15,18] run in transverse or frontal plane, hence they are called lateral curves of occlusion (Figure 1).

The roles of circumscribed Monson's pyramid in providing some insight into occlusal concepts and occlusion can be appreciated and understood in the following ways: The Monson's pyramid is specific with respect to individual's face size. The equilateral triangle which forms the base of the tetrahedron represents the Bonwill's equilateral triangle. The circumcircle of Bonwill's triangle represents a cut face of Monson's sphere. The circumcircle of the posterior triangle of the tetrahedron represents Monson's lateral or transverse occlusal curves.

The arcs of the lower segments of the circumcircles of the left and right surface equilateral triangles of the tetrahedron represent the sagittal occlusal curves of Spee; and the distance between chords and arcs of the segments represent the clinical heights of the crowns of maxillary posterior teeth and the combined depth of mandibular and maxillary sagittal occlusal curves of Spee. The three sides of Bonwill's equilateral triangle represent different chords on the circumsphere of the regular tetrahedron. The vertices of the tetrahedron correspond to important anthropometric points or landmarks in the study of dental occlusion and articulation i.e. the condylar processes, incisal point and glabella (Figure 1). If the posterior triangle of the tetrahedron is regarded as its base and the vertex (at the midpoint of the lower central incisors) represents its apex, then, a cone which adequately describes the conical theory of occlusion by Hall is formed (Figure 1).

The length of the perpendicular bisector of each of the four equilateral triangles, forming the faces of a regular tetrahedron, can be calculated using this formula H= a√3÷2 (12) where "H" is the length of the perpendicular bisector and "a" is the length of one side of the regular tetrahedron. The height of the regular tetrahedron can also be calculated using this formula h=√6x (a/3 (26) where "h" is the height and "a" is the length of the edge. The triangular faces, vertices and edges of the tetrahedron are similar to the morphological expressions of the basic elements of the maxillo-mandibulo-dental system. Consequently, Monson's pyramid can be used to explain common factors in the various elements of masticatory apparatus, articulation and occlusion theories [23-26].

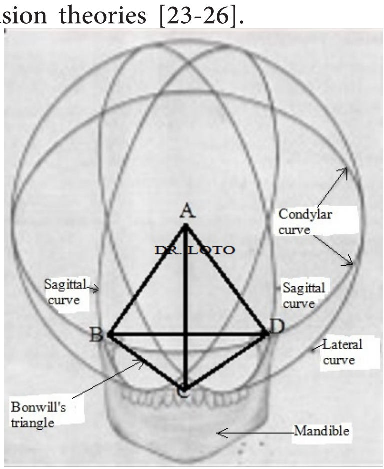

Figure 1. Showing formation of Monson's pyramid and sphere, Bonwill's triangle and compensating curves.

- A represents center of rotation of Monson's sphere-a 3D geometrical object.
- A, B, C and D represent the four vertices of Monson's pyramid which is derived from Bonwill's equilateral triangle.
- The four triangles that make up Monson's pyramid are BCD, ABC, ACD and ABD.
- Bonwill's equilateral triangle forms the base of Monson's pyramid.
- Monson's pyramid is a conical section of Monson's sphere with a circular cap.
- Sagittal curves represent Spee's sagittal occlusal curves.
- Lateral curve represents Wilson's lateral occlusal curve.
- The intersections of Spee and Wilson's curves form Monson's curves .
- Monson's pyramid could be considered as Hall's cone if C represents the apex and ABD represents the base of the pyramid.

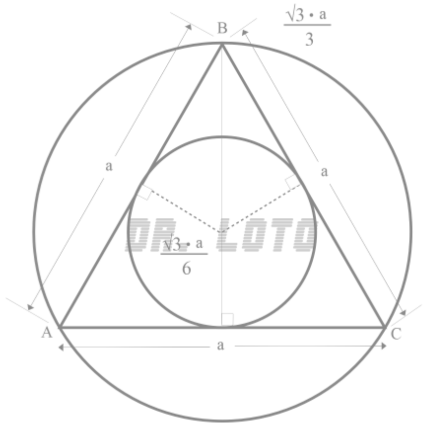

Figure 2. Bonwill's equilateral Triangle showing incircle and circumcircle.

- a = The length of each side of the triangle.
- The radius of the circumcircle is given by the formula √3×a÷3.
- The radius of the circumcircle is given by the formula √3×a÷6.
- A, B and C are the angles of the triangle = 60º each.

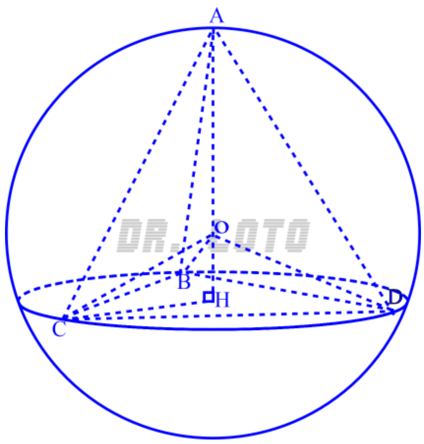

Figure 3. Showing a regular tetrahedron and its circumsphere.

- A, B, C and D represent the vertices of the tetrahedron.
- AH represents the height of the tetrahedron.
- O represents the center of the tetrahedron.
- ABC, ABD, ACD and BCD are the triangles that form the four surfaces of the tetrahedron.
- AC, AB, CB, CD, BD and AD are the edges of the tetrahedron.
- CB, CD and BD are cords formed by the base of the tetrahedron on a segment of the circumsphere of the tetrahedron.

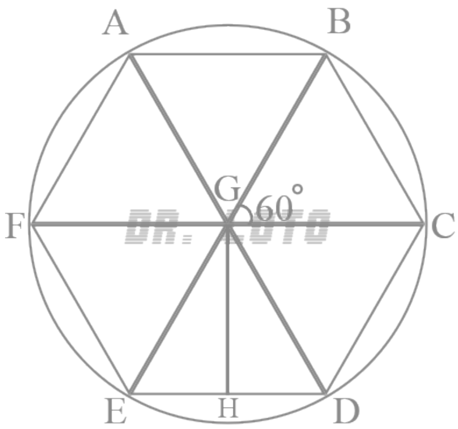

Figure 4. Showing six Bonwill's equilateral triangles within a common circumcircle.

- Each triangle subtends an angle of 60 degrees at the center.
- The length of each side of a triangle is 10cm.
- AB, BC, CD, DE, EF and FA are chords which divide the circumcircle into six equal arcs.

Determination of Hall's cone inclination

The inclination angle of Hall's cone is the angle between the long axis of the cone and the occlusal plane [7,8]. It is the sum of the angle between the long axis of the cone and its inferior surface and Balkwill angle. Balkwill angle is the angle between Bonwill equilateral triangle and the occlusal plane [28]. According to Dr. Balkwill, this angle is 26 degrees on the average, with a range of 22-30 degrees, in Caucasians [28]. Therefore, in determining the inclination angle of Hall's cone, the angle between the long axis and the inferior surface of the cone as well as Balkwill angle must be ascertained.

Consequently, if Balkwill angle (26 degrees) is added to the angle between the long axis of the cone and its inferior surface (19.47 degrees), a total of 45.47 degrees is obtained as the inclination angle of Hall's cone. This calculated angular value closely approximates Hall's inclination angle which is 45 degrees (Table 4). It should be noted that the angle between the long axis of the cone and its inferior surface remains constant [27]. However, Balkwill's angle varies from individual to individual and from race to race. Therefore, Hall's cone inclination angle will vary because of variations in Balkwill's angle. Studies of Bergstrom [29], Hart [30] and Kohler [31] have shown average values of Balkwill angle of 18 degrees, 20 degrees and 21 degrees respectively. To this end, it is important to carry out studies in different populations and in different geographical locations to determine average norms of Balkwill angle for different populations around the world.

A major limitation of this study was that Bonwill's equilateral triangle, which formed the basis of this analysis, could not be used universally because it was based on Caucasians; and studies have shown that there are variations in bicondylar distances from one person to another person and within a racial grouping as well as from one race to another race. However, this limitation can be eliminated by determining the bicondylar distance of each patient for the construction of a regular tetrahedron (Monson's pyramid), which is the unifying 3D geometric figure for all the aforementioned geometrical models of occlusion.

Conflict of interest

No conflict of interest is declared.

Conclusion

It is concluded as follows:

1. The hypothesis that Monson's pyramid, a regular tetrahedron derived from Bonwill's triangle, is the unifying 3D geometric figure for Bonwill's, Monson's, Spee's and Hall's theories of occlusion was established.

2. The hypothesis that Monson's pyramid constitutes one of the four regular tetrahedrons that make-up Monson's sphere was established.

3. The hypothesis that the radius of Monson's sphere is greater than the radius of circumsphere of Monson's pyramid was also established.

4. There is a strong mathematic-geometric interconnectedness among Bonwill's triangle, Spee's circle/curves, Monson's sphere/curves/pyramid and Hall's cone.

5. Bonwill's triangle, Spee's circle/curves, Monson's sphere/curves/pyramid and Hall's cone are specific for each patient based on the individual patient's bi-condylar distance.

6. The clinical significance of this study is that some important linear and angular parameters such as curved and straight inter-canine distances, depth of sagittal and lateral occlusal curves, anterior facial height and Balkwill angle, that are required in the fabrication of dentures, can be calculated from Monson's pyramid and its associated geometric figures.

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